

Pleijel's theorem for Schrödinger operators with radial potentials

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Abstract

In 1956, Å. Pleijel gave his celebrated theorem showing that the inequality in Courant's theorem on the number of nodal domains is strict for large eigenvalues of the Laplacian. This was a consequence of a stronger result giving an asymptotic upper bound for the number of nodal domains of the eigenfunction as the eigenvalue tends to $+\infty$. A similar question occurs naturally for the case of the Schrödinger operator. The first significant result has been obtained recently by the first author for the case of the harmonic oscillator. The purpose of this paper is to consider more general potentials which are radial. We will analyze either the case when the potential tends to $+\infty$ or the

case when the potential tends to zero, the considered eigenfunctions being associated with the eigenvalues below the essential spectrum.

1 Introduction

The goal of this paper is to extend Pleijel's theorem for the Dirichlet Laplacian $H(\Omega) = -\Delta$ in a bounded domain Ω to the case of the Schrödinger operator $H_V = -\Delta + V$ in \mathbb{R}^d . We are interested in counting the number of nodal domains of an eigenfunction and to relate this number with the labelling of the corresponding eigenvalue. Throughout this paper, for any function f defined over a domain $D \subset \mathbb{R}^d$, $\mu(f)$ will denote the number of nodal domains of f , namely the number of connected components of $D \setminus f^{-1}(0)$. The starting point of the analysis is Courant's Theorem (1923) [8].

Theorem 1.1 (Courant) *If ϕ_n is an eigenfunction associated with the n -th eigenvalue λ_n of $H(\Omega)$ (ordered in non decreasing order and labelled with multiplicity), then*

$$\mu(\phi_n) \leq n. \quad (1.1)$$

Pleijel's theorem (1956) [26] says

Theorem 1.2 (Pleijel's weak theorem) *If the dimension is ≥ 2 , there is only a finite number of eigenvalues of the Dirichlet Laplacian in Ω for which we have equality in (1.1).*

Let us now give the strong form of Pleijel's theorem.

Theorem 1.3 (Pleijel's strong theorem) *Let λ_n the non decreasing sequence of eigenvalues associated to the Dirichlet realization of the Laplacian. For any $d \geq 2$ for any orthonormal basis ϕ_n of eigenfunctions ϕ_n of $H(\Omega)$ (the Dirichlet Laplacian in Ω) associated with $\lambda_n(\Omega)$,*

$$\limsup_{n \rightarrow +\infty} \frac{\mu(\phi_n)}{n} \leq \gamma(d) = \frac{2^{d-2} d^2 \Gamma(d/2)^2}{(j_{\frac{d}{2}-1}^d)^d}, \quad (1.2)$$

where j_ν denotes the first zero of the Bessel function J_ν .

The theorem was proved by Pleijel [26] for $d = 2$ and then extended by Peetre [25] and Bérard-Meyer [3]. We recall from [3, Lemma 9] that Pleijel's

constant equals

$$\gamma(d) = w_d^{-1} \omega_d^{-1} (\lambda(B_d))^{-d/2} < 1, \quad (1.3)$$

where

- w_d is the Weyl constant

$$w_d := (2\pi)^{-d} \omega_d, \quad (1.4)$$

-

$$\omega_d := |B_d|, \quad (1.5)$$

where B_d is the unit ball in \mathbb{R}^d and $|D|$ denotes for an open set $D \subset \mathbb{R}^d$ its volume;

- $\lambda(B_d)$ is the Dirichlet ground state energy of the Laplacian in B_d .

As $\gamma(d) < 1$, one recovers as a corollary that the inequality in Courant's theorem is strict for n large. The second point to notice is that the constant is independent of the open set. Complementary properties of $\gamma(d)$ have been obtained by B. Helffer and M. Persson-Sundqvist [16]. In particular $d \mapsto \gamma(d)$ is decreasing exponentially to 0. Finally note that this constant is not optimal (see [4], [33] and the discussion in [15]).

The original proof of Pleijel's theorem is based on a combination between the Weyl formula and the Faber-Krahn inequality. Weyl's theorem reads, as $\lambda \rightarrow +\infty$,

$$N(\lambda) = |\Omega| w_d \lambda^{\frac{d}{2}} (1 + o(1)), \quad (1.6)$$

where

$$N(\lambda) := \#\{\lambda_j < \lambda\}. \quad (1.7)$$

The Faber-Krahn inequality reads

Theorem 1.4 *For any domain $D \subset \mathbb{R}^d$ ($d \geq 2$), we have*

$$|D|^{\frac{2}{d}} \lambda(D) \geq \omega_d^{\frac{2}{d}} \lambda(B_d). \quad (1.8)$$

There are a lot of Weyl's formulas available in the context of the Schrödinger operator $H_V := -\Delta + V$. The use of the Faber-Krahn inequality is more problematic, except of course for the case of bounded domains with bounded potential which can be treated like the membrane case. In 1989 Leydold [21] obtained in his diploma thesis a weak Pleijel theorem for the isotropic harmonic oscillator (see also [2]). Two years ago Charron [6] in his master thesis proved Pleijel's strong theorem also for the harmonic oscillator:

Theorem 1.5 (Charron's theorem) *Let $(\phi_n)_{n \in \mathbb{N}}$ be an orthogonal basis of eigenfunctions in $L^2(\mathbb{R}^d)$ of the harmonic oscillator associated with $(\lambda_n)_{n \in \mathbb{N}}$. Then*

$$\limsup_{n \rightarrow +\infty} \frac{\mu(\phi_n)}{n} \leq \gamma(d). \quad (1.9)$$

The theorem is also proven in the case of the non-isotropic harmonic oscillator [7]. The interesting fact is that the potential $V(x) = \sum_i a_i x_i^2$ (with $a_i > 0$) does not appear on the right hand side of the upper bound. Note also that when there are no eigenvalue degeneracies a much stronger result is available in [7].

A natural question to ask is whether the theorem can be extended to more general Schrödinger operators. We will answer positively this question under the additional assumption that the potential is radial. More precisely, we assume

$$d \geq 2$$

and we consider on \mathbb{R}^d a Schrödinger operator $H_V = -\Delta + V$, where V is a radial potential:

$$V(x) = v(r), \quad (1.10)$$

with $|x| = r$.

We will assume that

$$v \in C^1(0, +\infty), \quad (1.11)$$

and that there exists $R_0 > 0$ such that

$$v'(r) > 0, \text{ for } r \geq R_0. \quad (1.12)$$

In order to allow some singularity at the origin, we assume either

$$v \in C^0([0, +\infty)), \quad (1.13)$$

or that there exists $s \in (0, 2)$ such that, as $r \rightarrow 0$,

$$v(r) \approx -r^{-s}. \quad (1.14)$$

Here, we say that $a \approx b$ is a/b and b/a are bounded, i.e. if there exists $C > 0$ such that

$$\frac{1}{C} \leq \frac{a}{b} \leq C.$$

We will study two cases according to the behavior of v at $+\infty$.

Case A: v tends to $+\infty$ as $r \rightarrow +\infty$.

More precisely, we assume (1.11)-(1.14) and that there exists¹ $m > 1$ such that as $r \rightarrow +\infty$

$$v(r) \approx r^m, \quad (1.15)$$

and

$$v'(r) \approx r^{m-1}. \quad (1.16)$$

Case B: v tends to 0 as $r \rightarrow +\infty$.

More precisely, we assume (1.11)-(1.14) and that there exists $m \in (-2, 0)$ such that

$$v(r) \approx -r^m, \quad (1.17)$$

and

$$v'(r) \approx r^{m-1}. \quad (1.18)$$

In the two cases there is a natural selfadjoint extension starting from $C_0^\infty(\mathbb{R}^d)$ (see Section 3). In Case A, the spectrum is discrete and consists of a non decreasing sequence of eigenvalues λ_n tending to $+\infty$. In Case B the spectrum is divided in two parts, the essential spectrum: $[0, +\infty)$ and the discrete spectrum, which consists of an infinite sequence of negative eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ tending to 0 (see for example Reed-Simon [28], Vol. IV, Theorem XIII.6). Associated with this sequence $(\lambda_n)_{n \in \mathbb{N}}$, we can consider an orthonormal sequence of eigenfunctions ϕ_n , where in Case A ϕ_n is an Hilbertian basis of $L^2(\mathbb{R}^d)$ and in Case B of the negative eigenspace.

Our analysis will contain two well-known potentials: the quantum harmonic oscillator (Case A) and the Coulomb potential (Case B). In both cases, we know the eigenvalues and an explicit basis of eigenfunctions but in the proof this property will not be used. Our aim is to prove the following result:

Theorem 1.6 (Pleijel's theorem for Schrödinger) *In Cases A or B, if $(\phi_n)_{n \geq 1}$ is an orthogonal sequence of eigenfunctions of H_V associated with the above defined sequence λ_n , then*

$$\limsup_{n \rightarrow +\infty} \frac{\mu(\phi_n)}{n} \leq \gamma(d). \quad (1.19)$$

The paper is organized as follows. In Section 2, we discuss the general strategy and the methods used by Pleijel

¹This condition appears when applying Weyl's formula given by Theorem 4.2 from [28]. At least under stronger assumptions on the regularity of v for $r \rightarrow +\infty$, it should be possible to assume $m > 0$.

first and then by P. Charron. In Section 3, we review the general properties of the Schrödinger operator. In Section 4 we collect those Weyl-type results we need for the proof of Theorem 1.6.

In Section 5, we give the proof of our Pleijel's theorem in the two situations.

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2 About the methods

As recalled in the introduction, the original proof of Å. Pleijel was based on a tricky combination of Weyl's formula with the Faber-Krahn inequality. When considering the case of the Schrödinger operator in \mathbb{R}^d , Weyl's formula still exists but the use of Faber-Krahn is not easy: nodal domains could be unbounded and the variation of the potential inside a nodal domain could be very high. One has consequently to find an idea for proving that these two bad situations do not occur very often.

In the case of the harmonic oscillator Charron's proof relies on specific properties of the eigenfunctions and the potential. Namely, it used the fact that every eigenfunction is a linear combination of an exponential multiplied by polynomials whose degree can be controlled by a function of the labeling of the eigenvalue, that the hypersurfaces with constant potential are hyperspheres and the fact that the counting function $N(\lambda)$ behaves nicely as $\lambda \rightarrow +\infty$ (Weyl's law).

In addition, it also used that, every nodal domain of an eigenfunction of a Schrödinger operator intersects the classically allowed region associated with the eigenvalue λ , i.e

$$V^{(-1)}(-\infty, \lambda) := \{x \in \mathbb{R}^d \mid V(x) < \lambda\}. \quad (2.1)$$

This property is quite general and elementary.

The key was then to divide the classically allowed region in a finite number of annuli of the form $V^{(-1)}(a, b)$. Every nodal domain can either be contained in a single annulus or intersect more than one. To give an upper bound on

the number of nodal domains not contained in one annulus, Charron uses properties of algebraic surfaces, as well as results arising in Morse theory adapted from Milnor [23].

Then, he used the Faber-Krahn inequality to give a lower bound on the volume of any nodal domain contained in a single annulus. Dividing the volume of each annulus by the volume of each nodal domain gave an upper bound on the number of nodal domains contained in that annulus.

The last step was to find an appropriate number of annuli to balance out both estimates.

To extend the methods of Charron's proof to more general potentials, we need to find Schrödinger operators such that:

- (i) There are good lower bounds for the number of eigenvalues below any λ .
- (ii) We can count the number of nodal domains that intersect a given energy hypersurface $V^{(-1)}(b)$.
- (iii) We can give an upper bound on the number of nodal domains that are not contained in the classically allowed region.

In the case of the harmonic oscillator, the eigenvalues are known explicitly. However, for many potentials V , Weyl's law can be extended to the Schrödinger operator H_V for estimating the number of eigenvalues. Hence, we need to find a suitable class of potentials where this law holds.

So far, the only known method to give a suitable upper bound on the number of nodal domains that intersect an energy hypersurface are based on Milnor's theorem (see Subsection 3.7). Hence we need this hypersurface to be algebraic and the property that for any eigenvalue λ and any energy hypersurface (or at least a suitable family) the restriction of any associated eigenfunction u_λ equals the restriction of a polynomial to this hypersurface. This is why we focus in this paper on the study of radial potentials. In this case, the energy hypersurfaces are hyperspheres $\{r = \rho\}$ for some $\rho > 0$ and it can be shown that the restriction of an eigenfunction to a hypersphere is always a linear combination of hyperspherical harmonics, each one being the restriction to the hypersphere of a homogeneous harmonic polynomial. We will also have to control the degree of such polynomials by a function of λ or of its labelling. This last property will allow us to bound the number of nodal domains that are not contained in $V^{(-1)}(-\infty, \lambda)$.

Another problem might arise when estimating the number of nodal domains contained in one annulus. In the case of the harmonic oscillator, summing over all annuli gives us an expression which can be compared directly with an integral. The error term that arises becomes negligible as $\lambda \rightarrow +\infty$. It remains to show under which conditions on V the same method can be applied.

Finally, in the specific case of Coulomb-like potentials at the origin, we need to look at the behavior of the number of nodal domains near the origin.

3 On the spectral theory of the Schrödinger operators with radial potential

3.1 General theory

We first verify that our Schrödinger operator $H_V = -\Delta + V$ is well defined by a Friedrichs procedure starting from its sesquilinear form defined on $C_0^\infty(\mathbb{R}^d \setminus \{0\}) \times C_0^\infty(\mathbb{R}^d \setminus \{0\})$

$$(u, v) \mapsto a(u, v) := \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x) dx + \int_{\mathbb{R}^d} V(x) u(x) v(x) dx.$$

Note that the left term has a meaning as soon as $V \in L_{loc}^1(\mathbb{R}^d \setminus \{0\})$. In our case, this is a consequence of Assumption 1.11. Our operator, will be defined through a Friedrichs extension. This works as soon as $q(u) := a(u, u)$ is bounded from below by $-C \|u\|_{L^2}^2$. It is consequently enough to control the integral $\int_{V < 0} V(x) u(x)^2 dx$ from below.

When $d \geq 3$, we use Hardy's inequality

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \geq \left(\frac{d-2}{2} \right)^2 \int_{\mathbb{R}^d} \frac{1}{r^2} |u(x)|^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^d), \quad (3.1)$$

with $r = |x|$.

This inequality extends to $H^1(\mathbb{R}^d)$ by density.

For $d = 2$, we can use the modified Hardy inequality in a disk $D(0, \check{R})$ which reads (see for example [10])

$$\int_{\mathbb{R}^2} |\nabla u(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{u(x)^2}{|x|^2 \log^2(|x|/\check{R})} dx, \quad \forall u \in C_0^\infty(D(0, \check{R})), \quad (3.2)$$

which also extends to $H_0^1(D(0, \check{R}))$.

Using these inequalities and a partition of unity, the semi-boundedness on

$C_0^\infty(\mathbb{R}^d \setminus \{0\})$ follows immediately.

Let us now describe the form domain resulting from the Friedrichs extension procedure. We have:

Case A

$$Q_H = \{u \in H^1(\mathbb{R}^d) \mid \sqrt{V}u \in L^2(D(0, R_1)^c)\}$$

where R_1 is chosen such that $v(r) \geq 1$ for $r \in (R_1, +\infty)$,

Case B

$$Q_H = H^1(\mathbb{R}^d).$$

We do not need to characterize the domain of the corresponding self-adjoint operator.

3.2 Nodal domains intersect the classically allowed region

We use a similar argument as in [21] and [7]. We assume that we are either in Case A or in Case B, but the result is much more general.

Proposition 3.1 *Let λ be an eigenvalue below the essential spectrum, u_λ be an eigenfunction of H_V associated with eigenvalue λ and Ω be a nodal domain of u_λ . Then*

$$\Omega \cap V^{(-1)}(-\infty, \lambda) \neq \emptyset.$$

Proof.

If for all $x \in \Omega$, $V(x) > \lambda$, then

$$\begin{aligned} \lambda &= \frac{\int_\Omega |\nabla u_\lambda(x)|^2 dx + \int_\Omega V(x) u_\lambda(x)^2 dx}{\int_\Omega u_\lambda(x)^2 dx} \\ &\geq \frac{\int_\Omega V(x) u_\lambda(x)^2 dx}{\int_\Omega u_\lambda(x)^2 dx} \\ &> \frac{\int_\Omega \lambda u_\lambda(x)^2 dx}{\int_\Omega u_\lambda(x)^2 dx} = \lambda, \end{aligned} \quad (3.3)$$

hence a contradiction. \square

Therefore, any nodal domain is either contained in the classically allowed region $\{V < \lambda\}$ or intersects the hypersurface $V^{(-1)}(\lambda)$.

3.3 The radial Schrödinger operator

Although the exposition there is limited to the case $d = 3$, one can refer to Reed-Simon [28], Vol. IV p. 90-91.

The Laplace operator $-\Delta$ can be written as

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2}(-\Delta_{\mathbb{S}^{d-1}}), \quad (3.4)$$

where $r = |x|$ is the radial variable and $\Delta_{\mathbb{S}^{d-1}}$ is the Laplace-Beltrami operator, acting in $L^2(\mathbb{S}^{d-1})$. The following proposition is standard (see for example [31], Theorem 22.1 and Corollary 22.1).

Proposition 3.2 *Assume that $d \geq 2$. The spectrum of $-\Delta_{\mathbb{S}^{d-1}}$ consists of eigenvalues*

$$\ell(\ell + d - 2), \quad \ell \in \mathbb{N}.$$

The multiplicity of the eigenvalue $\ell(\ell + d - 2)$ is given by

$$\Lambda_{\ell,d} := \binom{\ell + d - 1}{d - 1} - \binom{\ell + d - 3}{d - 1},$$

which coincides with the dimension of the space of homogeneous, harmonic polynomials of degree ℓ .

We denote by $\mathbb{S}^{d-1} \ni \omega \mapsto Y_{\ell,m}(\omega)$ an orthonormal basis of the $\Lambda_{\ell,d}$ -dimensional eigenspace associated with $\ell(\ell + d - 2)$. We recall that each $Y_{\ell,m}$ is the restriction to \mathbb{S}^{d-1} of a harmonic homogeneous polynomial of degree ℓ .

We now consider the Schrödinger operator H_V and assume

$$V(x) = v(r). \quad (3.5)$$

In this case, one can determine the spectrum by using polar coordinates. In the spherical coordinates, we can determine the spectrum by considering the (closure of the) union of the spectra of the family (indexed by $\ell \in \mathbb{N}$) of Sturm Liouville operators \mathcal{L}_ℓ defined by

$$\mathcal{L}_\ell = -\frac{d^2}{dr^2} - \frac{d-1}{r} \frac{d}{dr} + \frac{\ell(\ell + d - 2)}{r^2} + v(r), \quad (3.6)$$

acting in $L^2((0, +\infty), r^{d-1} dr)$, with a suitable Dirichlet like condition at 0 (see Reed-Simon [28], p. 91, Proof of Lemma 1). Note that the "Dirichlet like" condition is expressed after the unitary transform $u \mapsto r^{\frac{d-1}{2}} u$ sending

$L^2((0, +\infty); r^{d-1}dr)$ onto $L^2((0, +\infty); dr)$ and becomes the standard Dirichlet condition for $\ell = 0$. When $\ell > 0$, no condition is given. The new operator is then:

$$\widehat{\mathcal{L}}_\ell = -\frac{d^2}{dr^2} + \frac{(\ell + \frac{d-1}{2})(\ell + \frac{d-1}{2} - 1)}{r^2} + v(r).$$

Proposition 3.3 *Let $H_V = -\Delta + V$, where $V(x) = v(r)$ satisfies either Case A or Case B.*

Any eigenvalue of $-\Delta + V$ is of the form

$$\lambda = \lambda_{n,\ell}, \quad (3.7)$$

where $\lambda_{n,\ell}$ is the n -th eigenvalue of \mathcal{L}_ℓ .

A corresponding basis of eigenfunctions has the form

$$u_{n,\ell,m}(r, \omega) = f_{n,\ell}(r) Y_{\ell,m}(\omega), \quad (3.8)$$

where $Y_{\ell,m}(\omega)$ denotes an orthonormal basis of (hyper)spherical harmonics.

We recall that these functions form a basis of $L^2(\mathbb{R}^d)$ in Case A (see [28]) or a basis of the negative eigenspace in Case B.

3.4 Courant's nodal theorem and nodal behavior of eigenfunctions.

For the analysis of potentials with singularities it is worth to ask under which condition one can prove Courant's theorem or describe the local nodal structure of an eigenfunction. Under our assumptions the only point is the control at the origin. Outside the origin, (1.11) implies that the potential is C^1 and the structure of the nodal set is well known.

Looking at the proof of Courant's theorem, the only thing we need is the unique continuation theorem, i.e. we need to show that if an eigenfunction u_λ is identically 0 in a non empty open set ω then it is zero in \mathbb{R}^d . The argument clearly works if there is only a singularity at 0, because $\omega \setminus \{0\}$ is an open set where u_λ vanishes identically. See [17] for more properties. We will show in the next subsection that no nodal domain is contained in a sufficiently small ball around the origin. Hence the counting of nodal domains can start outside this ball.

3.5 No nodal domains in a small ball

In this subsection, we show that under our assumptions the nodal domain cannot be contained in a small neighborhood of the origin. We will start with Case B which is easier.

3.5.1 Case B

We have the following statement:

Proposition 3.4

If $d \geq 2$ and in Case B there exists $r_d > 0$ such that, if $\lambda < 0$ is an eigenvalue and u_λ is a corresponding eigenfunction, there is no nodal domain ω of u_λ contained in $B(0, r_d)$.

Proof

We first deduce from Assumption (1.14), because $s < 2$, that:

- For $d \geq 3$ there exists r_d such that

$$v(r) + \frac{(d-2)^2}{4r^2} > 0, \quad (3.9)$$

for $r \in (0, r_d)$.

- For $d = 2$, there exists $r_2 > 0$ such that

$$v(r) + \frac{1}{4r^2 \ln^2(r/2r_2)} > 0, \quad (3.10)$$

for $r \in (0, r_2)$.

We now use the identity

$$\int_{\omega} |\nabla u_\lambda(x)|^2 dx + \int_{\omega} V(x) |u_\lambda(x)|^2 dx = \lambda \int_{\omega} |u_\lambda(x)|^2 dx,$$

and get because $\lambda < 0$

$$\int_{\omega} |\nabla u_\lambda(x)|^2 dx + \int_{\omega} V(x) |u_\lambda(x)|^2 dx < 0.$$

When $d \geq 3$, we use Hardy's inequality (3.1) (u_λ is extended in \mathbb{R}^d by 0 outside ω and this extension is in $H^1(\mathbb{R}^d)$) and get that

$$\int_{\omega} \left(V(x) + \frac{(d-2)^2}{4r^2} \right) |u_\lambda(x)|^2 dx < 0.$$

This contradicts (3.9) if $\omega \subset B(0, r_d)$.

When $d = 2$, we use the modified Hardy inequality (3.2) with $\check{R} = 2r_2$, (u_λ is extended by 0 outside ω in $D(0, \check{R})$ and this extension is in $H_0^1(D(0, \check{R}))$) and get a contradiction with (3.10).

□

3.5.2 Case A

In Case A, with singularities, there is some difficulty because we consider λ large. When $d \geq 3$, the previous proposition will be true in a ball whose radius is $r_d(\lambda) \approx \lambda^{-\frac{1}{2}}$. For $d = 2$, $r_2(\lambda)$ could be taken as $r_2(\lambda) \approx \lambda^{-\frac{1}{2}-\epsilon}$ for some $\epsilon > 0$. More precisely, we have

Proposition 3.5 *Under Assumption (1.14) and if $d \geq 3$, there exists a constant $c_d > 0$ which depends on V and d only and $\lambda_0 > 0$ such that, for $\lambda \geq \lambda_0$ and if u_λ denotes an eigenfunction of H_V , there are no nodal domains of u_λ contained in $B(0, c_d \lambda^{-1/2})$.*

If $d = 2$, for any $\epsilon > 0$, there exists $\lambda_\epsilon > 0$ and c_V that depends only on V such that, for $\lambda \geq \lambda_\epsilon$ and if u_λ denotes an eigenfunction of H_V , there are no nodal domains of u_λ contained in $B(0, c_V \lambda^{-1/2-\epsilon})$.

Proof

By (1.14), there exists $C > 0$ and $r_0 > 0$, such that $V > -Cr^{-s}$ for $0 < r \leq r_0$. As $s \in (0, 2)$, there exists λ_0 such that for $\lambda \geq \lambda_0$, there exists $r_d(\lambda) \sim \frac{d-2}{2} \lambda^{-\frac{1}{2}}$ such that

$$\frac{(d-2)^2}{4} r^{-2} - Cr^{-s} > \lambda, \forall r \in (0, r_d(\lambda)).$$

This implies

$$\frac{(d-2)^2}{4} r^{-2} + v(r) - \lambda > 0, \forall r \in (0, r_d(\lambda)) \text{ and } \lambda \geq \lambda_0.$$

The proof is achieved by taking $0 < c < \frac{d-2}{2}$ in the statement of the proposition and using the Hardy inequality as in the second part of the proof

of Proposition 3.4.

For the case $d = 2$, a not optimal $r_2(\lambda) = \lambda^{-\frac{1}{2}-\epsilon}$ (for some $\epsilon > 0$) together with the modified Hardy inequality do the job for λ_ϵ large enough.

3.6 Upper bound for the degree of the polynomials associated with the spherical harmonics

In this subsection, we prove the existence of a rather optimal upper bound on the degree ℓ of the polynomials associated with the spherical harmonics $Y_{\ell m}$ appearing in the decomposition of an eigenfunction u_λ .

Proposition 3.6

In Cases A or B, if λ is an eigenvalue of H_V such that $\lambda < \liminf_{r \rightarrow +\infty} v$ then there exists p_λ such that, for any associated eigenfunction u_λ and for any τ satisfying $\inf v < \tau \leq \lambda$, we can find a polynomial $\mathcal{P}_{\tau,\lambda}$ of d variables of degree at most p_λ such that on $V^{(-1)}(\tau)$ in \mathbb{R}^d the restriction of u_λ is equal to the restriction of $\mathcal{P}_{\tau,\lambda}$.

Moreover, p_λ satisfies

$$p_\lambda \leq \max\{\ell \mid \ell \geq 1 \text{ and } \inf_r \left(v(r) + \frac{\ell(\ell + d - 2)}{r^2} \right) < \lambda \}. \quad (3.11)$$

Proof.

For given λ , u_λ has the form (see (3.8))

$$u_\lambda = \sum_{\lambda=\lambda_{n,\ell}} c_{n,\ell,m} u_{n,\ell,m}.$$

If we restrict u_λ to the hypersphere of radius $r_\tau = v^{(-1)}(\tau)$, we get

$$u_\lambda(r_\tau, \omega) = \sum_{\lambda=\lambda_{n,\ell}} d_{\lambda,\tau,\ell,m} Y_{\ell,m}(\omega).$$

Considering the property of $Y_{\ell,m}$, we can choose for the proof of the proposition

$$\mathcal{P}_{\tau,\lambda} = \sum d_{\lambda,\tau,\ell,m} P_{\ell,m}, \quad (3.12)$$

where $P_{\ell,m}$ is the homogeneous harmonic polynomial of degree ℓ such that

$$(P_{\ell,m})_{/\{r=1\}} = Y_{\ell,m}.$$

It remains, in order to prove (3.11), to determine the highest $\ell \geq 1$ such that $\lambda_{n,\ell} = \lambda$.

By the minimax principle, we have

$$\lambda = \lambda_{n,\ell} \geq m_\ell := \inf \left(v(r) + \frac{\ell(\ell + d - 2)}{r^2} \right).$$

We have indeed

$$\lambda_{n,\ell} = \int_0^{+\infty} |f'_{n,\ell}(r)|^2 r^{d-1} dr + \int_0^{+\infty} \left(v(r) + \frac{\ell(\ell + d - 2)}{r^2} \right) |f_{n,\ell}(r)|^2 r^{d-1} dr.$$

□

The behavior of m_ℓ should be analyzed but note that our assumptions imply that $m_\ell > -\infty$. Furthermore $\ell \mapsto m_\ell$ is strictly increasing, so we can set $p_\lambda := [\check{p}_\lambda] + 1$, where \check{p}_λ is the solution of $\lambda = m_{\check{p}_\lambda}$ and $[x]$ means the integer part of x .

Application: Determination of an upper bound of p_λ .

Case A

We can assume that $\ell \geq 1$. This simply implies later a choice of $p_\lambda \geq 1$. If we consider $v(r) = c r^m$ as a model case for $m > 1$ and $c > 0$, the infimum is obtained when

$$c m r^{m-1} - 2 \frac{\ell(\ell + d - 2)}{r^3} = 0,$$

i.e.

$$r = \left(\frac{2\ell(\ell + d - 2)}{c m} \right)^{\frac{1}{m+2}}.$$

So we get

$$\begin{aligned} \inf \left(r^m + \frac{\ell(\ell+d-2)}{r^2} \right) &= \left(\frac{2\ell(\ell+d-2)}{c m} \right)^{\frac{m}{m+2}} + \ell(\ell + d - 2) \left(\frac{2\ell(\ell+d-2)}{c m} \right)^{-\frac{2}{m+2}} \\ &= \left(\frac{2}{c m} \right)^{\frac{m}{m+2}} \frac{c m + 2}{2} (\ell(\ell + d - 2))^{\frac{m}{m+2}}. \end{aligned}$$

This gives us

$$\check{p}_\lambda \sim a_m \lambda^{\frac{m+2}{2m}}, \quad (3.13)$$

with

$$a_m = \left(\frac{2}{c m} \right)^{-\frac{1}{2}} \left(\frac{c m + 2}{2} \right)^{-\frac{2m}{m+2}}.$$

For $m = 2$, we recover what we got for the harmonic oscillator by direct computation.

To treat the general case, we use the lower bound:

$$v(r) \geq c r^m - \frac{C}{r^s}. \quad (3.14)$$

We have to estimate

$$\inf \left(c r^m - \frac{C}{r^s} + \frac{\ell(\ell + d - 2)}{r^2} \right).$$

We observe that:

$$\begin{aligned} & \inf \left(c r^m - \frac{C}{r^s} + \frac{\ell(\ell + d - 2)}{r^2} \right) \\ & \geq \inf \left(c r^m + \frac{\ell(\ell + d - 2)}{2r^2} \right) + \inf \left(-\frac{C}{r^s} + \frac{\ell(\ell + d - 2)}{2r^2} \right). \end{aligned}$$

But, there exists (see below the computation in (3.17) with $m = -s$) a constant $C_0 > 0$, such that, for all $\ell \geq 1$,

$$\inf \left(-\frac{C}{r^s} + \frac{\ell(\ell + d - 2)}{2r^2} \right) \geq -C_0,$$

and we can use the lower bound of the model case above to get:

$$\inf_r \left(v(r) + \frac{\ell(\ell + d - 2)}{r^2} \right) \geq \frac{1}{2} \left(\left(\frac{2}{cm} \right)^{\frac{m}{m+2}} \frac{cm + 2}{2} (\ell(\ell + d - 2))^{\frac{m}{m+2}} \right) - C_0.$$

Hence we obtain like for the model case:

Corollary 3.7 *In Case A, as $\lambda \rightarrow +\infty$,*

$$p_\lambda \approx \lambda^{\frac{m+2}{2m}}. \quad (3.15)$$

Case B

Let us now compute an example corresponding to Case B. If we take $v(r) = -r^m$ for $m \in (-2, 0)$, the infimum is obtained when

$$mr^{m-1} + 2\frac{\ell(\ell + d - 2)}{r^3} = 0.$$

i.e.

$$r = \left(\frac{2\ell(\ell + d - 2)}{-m} \right)^{\frac{1}{m+2}}. \quad (3.16)$$

So we get

$$\begin{aligned} \inf \left(-r^m + \frac{\ell(\ell+d-2)}{r^2} \right) &= - \left(\frac{2\ell(\ell+d-2)}{-m} \right)^{\frac{m}{m+2}} + \ell(\ell+d-2) \left(\frac{2\ell(\ell+d-2)}{-m} \right)^{-\frac{2}{m+2}} \\ &= - \left(\frac{2}{-m} \right)^{\frac{m}{m+2}} \frac{m+2}{2} (\ell(\ell+d-2))^{\frac{m}{m+2}}. \end{aligned} \quad (3.17)$$

This gives us for $\lambda \rightarrow 0$, $\lambda > 0$,

$$\check{p}_\lambda \sim a_m(-\lambda)^{\frac{m+2}{2m}}, \quad (3.18)$$

with

$$a_m = \left(\frac{2}{-m} \right)^{-\frac{1}{2}} \left(\frac{m+2}{2} \right)^{-\frac{2m}{m+2}}. \quad (3.19)$$

In the Coulomb case $m = -1$ and $d = 3$, we get

$$\check{p}_\lambda \sim a_{-1}(-\lambda)^{-\frac{1}{2}}, \quad (3.20)$$

to compare with the direct computation which can be done for the Coulomb case.

In the general case, we can use

$$v(r) \geq -C r^{-s}, \quad \forall r \in (0, R)$$

and

$$v(r) \geq -c r^m, \quad \forall r \in (R, +\infty).$$

We will use twice the analysis of the model, the first time with m replaced by $-s$.

Let us first consider

$$\inf_{r \in (0, R)} \left(v(r) + \frac{\ell(\ell+d-2)}{r^2} \right) \geq \inf_{r \in (0, R)} \left(-C r^{-s} + \frac{\ell(\ell+d-2)}{r^2} \right).$$

We observe (see (3.16) with $m = -s$) that for ℓ large enough the map $r \mapsto -C r^{-s} + \frac{\ell(\ell+d-2)}{r^2}$ is decreasing on $(0, R)$. Hence

$$\inf_{r \in (0, R)} \left(-C r^{-s} + \frac{\ell(\ell+d-2)}{r^2} \right) = \left(-C R^{-s} + \frac{\ell(\ell+d-2)}{R^2} \right) \geq -C_s.$$

For the second case, we can use

$$\inf_{r \in (R, +\infty)} \left(-C r^m + \frac{\ell(\ell+d-2)}{r^2} \right) \geq \inf_{r \in (0, +\infty)} \left(-C r^m + \frac{\ell(\ell+d-2)}{r^2} \right),$$

and what we obtained for the homogeneous model.

Corollary 3.8 *In Case B,*

$$p_\lambda \approx (-\lambda)^{\frac{m+2}{2m}}. \quad (3.21)$$

3.7 Nodal domains on hyperspheres

Since the considered potentials V are radial, the energy hypersurfaces $\{V = \alpha\lambda\}$ are hyperspheres centered at 0. Also, the restriction of any eigenfunction u_λ of H_V to a hypersphere equals the restriction of some harmonic polynomial. We can use the following result proven in [7], which is based on [23]:

Proposition 3.9 *Let P be a polynomial of degree k with d variables. Then its restriction to the hypersphere \mathbb{S}^{d-1} admits at most $2^{2d-1}k^{d-1}$ nodal domains.*

We will combine this with the previous estimates obtained in Corollaries 3.7 and 3.8 to obtain an upper bound on the number of nodal domains on any hypersphere.

4 Weyl's formula

4.1 Preliminaries

For Schrödinger operators, Weyl's formula takes (under of course suitable assumptions to be discussed below) the form

$$N(\lambda) \sim (2\pi)^{-d} \int_{\xi^2 + V(x) \leq \lambda} dx d\xi. \quad (4.1)$$

After integration in the ξ variable, we get

$$N(\lambda) \sim W(\lambda), \quad (4.2)$$

where

$$W(\lambda) := w_d \int (\lambda - V)_+^{\frac{d}{2}} dx, \quad (4.3)$$

with w_d defined in (1.4).

This formula makes sense in case A (as $\lambda \rightarrow +\infty$) and in case B (as $\lambda \rightarrow 0$ with $\lambda < 0$). Let us just compute the right hand side for the two toy models: the harmonic oscillator and the Schrödinger operator with Coulomb potential. For the harmonic oscillator, we simply get

$$W(\lambda) = w_d \int (\lambda - r^2)_+^{\frac{d}{2}} dx = h_d w_d \lambda^d, \quad (4.4)$$

with

$$h_d := \int (1 - r^2)_+^{\frac{d}{2}} dx > 0.$$

More generally, if $v(r) = r^m$ for $m > 0$, we obtain, as $\lambda \rightarrow +\infty$,

$$W(\lambda) = w_d h_{d,m} \lambda^{d(\frac{1}{2} + \frac{1}{m})}. \quad (4.5)$$

In the Coulomb case, we get, with $\lambda < 0$

$$W(\lambda) = w_d \int \left(\lambda + \frac{1}{r} \right)_+^{\frac{d}{2}} dx = e_d w_d (-\lambda)^{-\frac{d}{2}}, \quad (4.6)$$

with

$$e_d := \int \left(\frac{1}{|x|} - 1 \right)_+^{\frac{d}{2}} dx = |\mathbb{S}^{d-1}| \int_0^1 (1 - r)^{\frac{d}{2}} r^{\frac{d}{2}-1} dr < +\infty. \quad (4.7)$$

More generally, if $v(r) = -r^m$, for $m \in (-2, 0)$, we obtain as $\lambda \rightarrow 0$ ($\lambda < 0$),

$$W(\lambda) = w_d h_{d,m} |\lambda|^{d(\frac{1}{2} + \frac{1}{m})}. \quad (4.8)$$

Observing that $N(\lambda_n) = n - 1$ if $\lambda_{n-1} < \lambda_n$, and assuming that the Weyl formula is proven (see below for the proof), we get conversly

$$\lambda_n \sim \check{w}_d n^{\frac{1}{d}} \quad \text{with } 1 = h_d w_d (\check{w}_d)^d, \quad (4.9)$$

in the case of the harmonic oscillator and

$$\lambda_n \sim -\check{v}_3 n^{-\frac{2}{3}}, \quad \text{with } 1 = e_3 w_3 (\check{v}_3)^{-\frac{3}{2}}, \quad (4.10)$$

in the case of the Coulomb case.

More generally we have the proposition:

Proposition 4.1 *In Cases A or B*

$$W(\lambda) \approx |\lambda|^{d(\frac{1}{2} + \frac{1}{m})}, \quad (4.11)$$

where the asymptotics is as $\lambda \rightarrow +\infty$ in Case A and as $\lambda \rightarrow 0$ ($\lambda < 0$) in Case B.

Proof

Outside a ball we can use for estimating the integral defining $W(\lambda)$ the comparison of $v(r)$ with r^m and then use the previous computations for the models. The control of the integral in a ball will be done in Subsection 4.3. \square

4.2 Weyl's formula under weak assumptions

There is vast literature on this subject: Reed-Simon [28] and references therein (for the historic), D. Robert [29], H. Tamura [34], Tulovski-Shubin [32], R. Beals [1], L. Hörmander [18, 19], A. Mohamed [24]. In the recent contributions the goal is to control the remainder but this is not important in the applications considered here. Here, we prefer to work under weaker assumptions and can use Theorem XIII.81 in Reed-Simon (Vol. 4) [28] for the case $V \rightarrow +\infty$ with a condition $d \geq 2$ and $m > 1$, and for the case $V \rightarrow 0$ as $|x| \rightarrow +\infty$, Theorem XIII.82. The treatment of the singularity is also explained (without detail) (see the discussion p. 277, lines -7 to -1, sending to Problem 132 therein). The idea there is to first prove a statement with V continuous and then to show that the addition of a potential W with compact support or in $L^{\frac{d}{2}}$ ($d \geq 3$) does not change the Weyl asymptotics.

Theorem XIII.81 in [28] reads:

Theorem 4.2 *Let V be a measurable function on \mathbb{R}^d ($d \geq 2$) obeying*

$$c_1 (r^\beta - 1) \leq V(x) \leq c_2 (r^\beta + 1), \quad (4.12)$$

and

$$|V(x) - V(y)| \leq c_3 [\max\{|x|, |y|\}]^{\beta-1} |x - y|, \quad (4.13)$$

for some $\beta > 1$ and suitable constants $c_1, c_2, c_3 > 0$.

Then

$$\lim_{\lambda \rightarrow +\infty} N(\lambda)/W(\lambda) = 1.$$

Remark 4.3 *The theorem is still true if we consider the Dirichlet problem for H_V in $\mathbb{R}^d \setminus B$, where B is a ball centered at 0. In Case A, the assumptions of the theorem are satisfied in $\mathbb{R}^d \setminus B$. This follows of our assumption (1.16) on v' .*

For Case B, Theorem XIII.82 in [28] reads:

Theorem 4.4 *Let V be a measurable function on \mathbb{R}^d ($d \geq 2$) obeying*

$$-c_1(r+1)^{-\beta} \leq V(x) \leq -c_2(r+1)^{-\beta}, \quad (4.14)$$

and

$$|V(x) - V(y)| \leq c_3 [1 + \min\{|x|, |y|\}]^{-\beta-1} |x - y|, \quad (4.15)$$

for some $\beta < 2$ and suitable constants $c_1, c_2, c_3 > 0$.

Then

$$\lim_{\lambda \rightarrow +\infty} N(\lambda)/W(\lambda) = 1.$$

Remark 4.5 *The theorem is still true if we consider the Dirichlet problem for H_V in $\mathbb{R}^d \setminus B$, where B is a ball centered at 0. In Case B, the assumptions of the theorem are satisfied in $\mathbb{R}^d \setminus B$. This is a consequence of our assumption on v' in (1.18).*

4.3 Treatment of the singularity

To cover the question of the treatment of the singularity at the origin we could think of using (Problem 132 in [28]) to treat the singularity as a perturbation. Due to the use of the Cwickel-Lieb-Rozenblum inequality [11, 22, 30] in the argument, this approach works only under the condition $d \geq 3$. If we remember that we only need a lower bound for $N(\lambda)$ one can proceed for $d \geq 2$ in the following way. We can introduce a small ball $B = B(0, \epsilon)$ around the singularity and look at the Dirichlet problem in $\mathbb{R}^d \setminus B$. We denote by $N_B(\lambda)$ the corresponding counting function. Because the eigenvalues are greater than the initial problem by monotonicity of the domain, the estimate of the $N(\lambda)$ of the new problem will give the lower bound:

$$N_B(\lambda) \leq N(\lambda).$$

The theorem in Reed-Simon [28] can be applied in $\mathbb{R}^d \setminus B$ (proof unchanged) and we get by Weyl's formula

$$N_B(\lambda) \sim W_B(\lambda),$$

with

$$W_B(\lambda) = w_d \int_{r \geq \epsilon} (\lambda - V)_+^{\frac{d}{2}} dx.$$

It remains to compare $W_B(\lambda)$ and $W(\lambda)$ in our two cases.

Case B

Here $\lambda < 0$. It is enough to show that, for some $\epsilon > 0$,

$$\int_{B(0, \epsilon)} (\lambda - V)_+^{\frac{d}{2}} dx < +\infty.$$

We have

$$\int_{B(0, \epsilon)} (\lambda - V)_+^{\frac{d}{2}} dx \leq C \int_0^\epsilon r^{d-1-s\frac{d}{2}} dr < +\infty,$$

the finiteness resulting from the assumption $s < 2$.

Case A.

Here $\lambda \geq \lambda_0 > 0$. We will show that $\int_{B(0, \epsilon)} (\lambda - V)_+^{\frac{d}{2}} dx$ is relatively small in comparison with $N(\lambda)$. In Case A, we have seen that

$$W(\lambda) \approx \int (\lambda - r^m)_+^{\frac{d}{2}} r^{d-1} dr \approx \lambda^{\frac{d}{2} + \frac{d}{m}}, \quad (4.16)$$

We have

$$\int_{B(0, \epsilon)} (\lambda - V)_+^{\frac{d}{2}} dx \leq C \int_0^\epsilon (\lambda + r^{-s})^{\frac{d}{2}} r^{d-1} dr \leq \hat{C} (\hat{C} + \lambda^{\frac{d}{2}}).$$

This gives, as $\lambda \rightarrow +\infty$,

$$\int_{B(0, \epsilon)} (\lambda - V)_+^{\frac{d}{2}} dx / W(\lambda) = \mathcal{O}(\lambda^{-\frac{d}{m}}).$$

Hence in the two cases, we have shown that $W_A(\lambda) \sim W(\lambda)$. In conclusion, we have obtained the following theorem.

Theorem 4.6

In Cases A or B, if $d \geq 2$, we have

$$N(\lambda) \geq W(\lambda)(1 + o(1)), \quad (4.17)$$

where the remainder $o(1)$ is as $\lambda \rightarrow +\infty$ in Case A and as $\lambda \rightarrow 0$ ($\lambda < 0$) in Case B.

5 Counting nodal domains

5.1 Preliminaries

We construct a radial partition of $\{V < \lambda\}$ of cardinality $\nu(\lambda)$ with $\nu(\lambda)$ to be defined later. When v is increasing on $(0, +\infty)$, $r_\lambda := v^{-1}(\lambda)$ is well defined in $(\inf v, +\infty)$ in Case A, and for any λ in $(-\infty, 0)$ in Case B. But we will only be interested $\lambda \rightarrow +\infty$ in Case A, and in $\lambda \rightarrow 0$ ($\lambda < 0$) in Case B. Under the weaker Assumption (1.12), we can define in the two cases r_λ by

$$r_\lambda := \sup_{v(r)=\lambda} r,$$

and obtain the asymptotics

$$r_\lambda \approx |\lambda|^{\frac{1}{m}}.$$

Note that in both cases r_λ tends to $+\infty$.

5.2 Analysis of Case A

5.2.1 A first partition of the classical region.

We recall that in this case, we assume $\lambda \geq \lambda_0 > 0$.

We introduce a partition of the classical region by introducing $\nu(\lambda)$ annuli, for $i = 1, \dots, \nu(\lambda)$,

$$D_i(\lambda) := \left\{ x \in \mathbb{R}^d \mid \left(\frac{i-1}{\nu(\lambda)} \right)^{1/d} r_\lambda < r < \left(\frac{i}{\nu(\lambda)} \right)^{1/d} r_\lambda \right\}.$$

We note that each annulus has the same volume:

$$|D_i(\lambda)| = \omega_d \frac{1}{\nu(\lambda)} r_\lambda^d \approx \frac{\lambda^{\frac{d}{m}}}{\nu(\lambda)}. \quad (5.1)$$

The cardinality $\nu(\lambda)$ satisfies a priori the condition

$$\lim_{\lambda \rightarrow +\infty} \nu(\lambda) = +\infty, \quad (5.2)$$

but the condition

$$\lim_{\lambda \rightarrow +\infty} \nu(\lambda)^{-1/d} r_\lambda = +\infty, \quad (5.3)$$

will appear along the proof.

The determination of λ_0 "large enough" will be given during the proof. If u_λ denotes some eigenfunction, we denote by $\mathcal{D}(u_\lambda)$ the set of the nodal domains of u_λ . We now introduce in $\mathcal{D}(u_\lambda)$ the following subsets.

Definition 5.1

$$A_i(u_\lambda) = \{\Omega \in \mathcal{D}(u_\lambda) \mid \Omega \subset D_i(\lambda)\} .$$

Here, i can take the values $1, 2, \dots, \nu(\lambda)$.

Definition 5.2

$$B_j(u_\lambda) = \left\{ \Omega \in \mathcal{D}(u_\lambda) \mid \Omega \cap \left\{ r = \left(\frac{j}{\nu(\lambda)} \right)^{1/d} r_\lambda \right\} \neq \emptyset \right\} .$$

Here, j can take the values $1, 2, \dots, \nu(\lambda)$.

From Subsection 3.2, we know that every nodal domain is contained in at least one of these sets.

Remark 5.3 *This partition will be refined by introducing, in the case of a singularity at the origin, a further partition of $A_1(u_\lambda)$.*

5.2.2 Counting the nodal domains contained in one annulus of the partition

We first count in each of the annuli $D_i(\lambda)$ for $i \geq 2$. Except if there are no singularity, the treatment of $A_1(u_\lambda)$ will be done separately. Hence we first prove the

Proposition 5.4

In Case A, if $\nu(\lambda)$ satisfies (5.2) and (5.3), we have the following inequality, as $\lambda \rightarrow +\infty$,

$$\sum_{i=2}^{\nu(\lambda)} \#A_i(u_\lambda) \leq \gamma(d) W(\lambda) \left(1 + \mathcal{O} \left(\frac{1}{\nu(\lambda)} \right) \right) . \quad (5.4)$$

Proof

If Ω is a bounded nodal domain of u_λ , the Faber-Krahn inequality (see Theorem 1.4) gives:

$$\frac{\int_{\Omega} |\nabla u_\lambda(x)|^2 dx}{\int_{\Omega} u_\lambda(x)^2 dx} \geq \left(\frac{1}{|\Omega|} \right)^{\frac{2}{d}} \omega_d^{\frac{2}{d}} \lambda(B_d), \quad (5.5)$$

where we recall that $|\Omega|$ denotes the volume of Ω .

We know that for a given bounded nodal domain Ω , we have

$$\lambda = \frac{\int_{\Omega} |\nabla u_\lambda|^2 dx + \int_{\Omega} V u_\lambda^2 dx}{\int_{\Omega} u_\lambda^2 dx},$$

which implies

$$\frac{\int_{\Omega} |\nabla u_\lambda|^2 dx}{\int_{\Omega} u_\lambda^2 dx} < \lambda - \inf_{x \in \Omega} V(x). \quad (5.6)$$

For all $\Omega \in A_i(u_\lambda)$, we obtain that

$$\frac{\int_{\Omega} |\nabla u_\lambda|^2 dx}{\int_{\Omega} u_\lambda^2 dx} < \lambda - v \left(\left(\frac{i-1}{\nu(\lambda)} \right)^{1/d} r_\lambda \right). \quad (5.7)$$

Because $i \geq 2$, we can, under Condition (5.3), combine (5.5) and (5.7) and obtain

$$|\Omega| \geq \frac{\omega_d \lambda(B_d)^{\frac{d}{2}}}{\left(\lambda - v \left(\left(\frac{i-1}{\nu(\lambda)} \right)^{1/d} r_\lambda \right) \right)^{\frac{d}{2}}}. \quad (5.8)$$

Observing that

$$\sum_{\Omega \in A_i(u_\lambda)} |\Omega| \leq |D_i(\lambda)|,$$

we obtain that

$$\# A_i(u_\lambda) \leq \frac{1}{\omega_d \lambda(B_d)^{\frac{d}{2}}} \left(|D_i(\lambda)| \left(\lambda - v \left(\left(\frac{i-1}{\nu(\lambda)} \right)^{1/d} r_\lambda \right) \right)^{\frac{d}{2}} \right). \quad (5.9)$$

Summing up for $i = 2, \dots, \nu(\lambda)$, we get

$$\sum_{i=2}^{\nu(\lambda)} \# A_i(u_\lambda) \leq \frac{1}{\omega_d \lambda(B_d)^{\frac{d}{2}}} \left(\sum_{i=2}^{\nu(\lambda)} |D_i(\lambda)| \left(\lambda - v \left(\left(\frac{i-1}{\nu(\lambda)} \right)^{1/d} r_\lambda \right) \right)^{\frac{d}{2}} \right). \quad (5.10)$$

We recognize on the right hand side a Riemann sum for the function $x \mapsto (\lambda - V(x))^{\frac{d}{2}}$ in $D(0, r_\lambda) \setminus (D_{\nu(\lambda)}(\lambda) \cup D_1(\lambda))$.

More precisely, we can write, using the monotonicity of v ,

$$\begin{aligned} \sum_{i=2}^{\nu(\lambda)} |D_i(\lambda)| \left(\lambda - v \left(\left(\frac{i-1}{\nu(\lambda)} \right)^{1/d} r_\lambda \right) \right)^{\frac{d}{2}} \\ \leq |D_2(\lambda)| \left(\lambda - v \left(\left(\frac{1}{\nu(\lambda)} \right)^{1/d} r_\lambda \right) \right)^{\frac{d}{2}} + \int (\lambda - V(x))^{\frac{d}{2}}_+ dx \\ \leq |D_2(\lambda)| \lambda^{\frac{d}{2}} + \int (\lambda - V(x))^{\frac{d}{2}}_+ dx. \end{aligned}$$

Using the asymptotic behavior of v at $+\infty$ given in (1.17), the computation of $\gamma(d)$ in (1.3), the asymptotic behavior of $W(\lambda)$ given in (4.11), and (5.1), we achieve the proof of Proposition 5.4.

5.2.3 Counting the nodal sets meeting the boundary of the annuli

Let us now turn to the study of the sets $B_i(u_\lambda)$. We have shown in Corollary 3.7 that $p_\lambda \approx \lambda^{\frac{m+2}{2m}}$. Using Proposition 3.9, we obtain that the number of nodal domains in a given $B_i(u_\lambda)$ satisfies

$$\#B_i(u_\lambda) \leq 2^{2d-1} p_\lambda^{d-1} \leq C_d \lambda^{\frac{(d-1)(m+2)}{2m}}. \quad (5.11)$$

Comparing with (4.11), we get that, for some $C > 0$ and $\lambda \geq \lambda_0$,

$$\sum_i \#B_i(u_\lambda) \leq C \nu(\lambda) \lambda^{-\frac{m+2}{2m}} W(\lambda).$$

If $\nu(\lambda)$ satisfies in addition,

$$\lim_{\lambda \rightarrow +\infty} \nu(\lambda) \lambda^{-\frac{m+2}{2m}} = 0, \quad (5.12)$$

we obtain

$$\lim_{\lambda \rightarrow +\infty} \frac{\sum_i \#B_i(u_\lambda)}{W(\lambda)} = 0. \quad (5.13)$$

5.2.4 Counting in $D_1(\lambda)$

We still have to consider when there is a singularity around 0. We treat first the case $d \geq 3$. We know from Proposition 3.5 that we can replace $D_1(\lambda)$ by the annulus $\widehat{D}_1(\lambda, C)$ defined by

$$\widehat{D}_1(\lambda, C) := \{x \in \mathbb{R}^d \mid \frac{1}{C} \lambda^{-\frac{1}{2}} < r < r_\lambda / \nu(\lambda)^{\frac{1}{d}}\}$$

for a sufficiently large C .

The number of nodal domains $\mu_{10}(\lambda)$ crossing the hypersphere $\{r = \frac{1}{C}\lambda^{-\frac{1}{2}}\}$ is controlled by (5.11):

$$\lim_{\lambda \rightarrow +\infty} \frac{\mu_{10}(\lambda)}{W(\lambda)} = 0. \quad (5.14)$$

To continue, we consider a partition of $\widehat{D}_1(\lambda)$ in two annuli:

$$D_{11}(\lambda) := \{x \in \mathbb{R}^d \mid \frac{1}{C}\lambda^{-\frac{1}{2}} < r < C\} \text{ and } D_{12}(\lambda) := \{x \in \mathbb{R}^d \mid C < r < r_\lambda/\nu(\lambda)^{\frac{1}{d}}\},$$

where we keep the liberty to choose $C > 0$ larger than the previous one.

Again the number $\mu_{11}(\lambda)$ of nodal domains crossing the hypersphere $\{r = C\}$ is controlled by (5.11):

$$\lim_{\lambda \rightarrow +\infty} \frac{\mu_{11}(\lambda)}{W(\lambda)} = 0. \quad (5.15)$$

Control in $D_{12}(\lambda)$.

The treatment of $D_{12}(\lambda)$ can be done for C large enough (in order to have the monotonicity of v) like the analysis of the $D_i(\lambda)$ for $i \geq 1$. More precisely, we can replace (5.9) (for $i = 1$) by

$$\# A_{12}(u_\lambda) \leq \frac{1}{\omega_d \lambda (B_d)^{\frac{d}{2}}} |D_{12}(\lambda)| (\lambda - v(C))^{\frac{d}{2}}, \quad (5.16)$$

where

$$A_{12}(u_\lambda) := \{\Omega \in \mathcal{D}(u_\lambda) \mid \Omega \subset D_{12}(\lambda)\}.$$

Hence we get, for some constant $C_d > 0$,

$$\# A_{12}(u_\lambda) \leq C_d \nu(\lambda)^{-1} \lambda^{\frac{d}{2} + \frac{d}{m}}, \quad (5.17)$$

which implies

$$\lim_{\lambda \rightarrow +\infty} (\# A_{12}(u_\lambda)/W(\lambda)) = 0. \quad (5.18)$$

Control in $D_{11}(\lambda)$.

Note that in $D_{11}(\lambda)$, we have, for some constant $C_s > 0$,

$$V(x) \geq -C_s \lambda^{\frac{s}{2}}, \quad \forall x \in D_{11}(\lambda).$$

Hence, for $\lambda \geq \lambda_0$, we deduce from (5.6), that

$$\frac{\int_\Omega |\nabla u_\lambda(x)|^2 dx}{\int_\Omega u_\lambda(x)^2 dx} \leq 2\lambda.$$

As in the proof of (5.8) we obtain:

$$|\Omega| \geq \omega_d \lambda (B_d)^{\frac{d}{2}} 2^{-\frac{d}{2}} \lambda^{-\frac{d}{2}}. \quad (5.19)$$

Summing over the Ω 's contained in $D_{11}(\lambda)$ and observing that the volume of $D_{11}(\lambda)$ is bounded, we get the existence of a constant \hat{C}_d such that

$$\#A_{11}(u_\lambda) \leq \hat{C}_d \lambda^{\frac{d}{2}},$$

where

$$A_{11}(u_\lambda) := \{\Omega \in \mathcal{D}(u_\lambda) \mid \Omega \subset D_{11}(\lambda)\}.$$

In particular we get

$$\lim_{\lambda \rightarrow +\infty} \frac{\#A_{11}(u_\lambda)}{W(\lambda)} = 0. \quad (5.20)$$

The case when $d = 2$ does not lead to new difficulties.

5.2.5 Conclusion for Case A

Summing all the upper bounds and having chosen $\nu(\lambda)$ satisfying (5.2), (5.3), and (5.12), we get, as $\lambda \rightarrow +\infty$,

$$\mu(u_\lambda) \leq \gamma(d) W(\lambda) (1 + o(1)). \quad (5.21)$$

Using the asymptotic upper bound (4.17), we get, as $\lambda \rightarrow +\infty$,

$$\mu(u_\lambda) \leq \gamma(d) N(\lambda) (1 + o(1)). \quad (5.22)$$

Observing that $N(\lambda_n) \leq n - 1$, we obtain Theorem 1.6.

5.3 Case B

The proof in case B follows the same lines. We define the sets $D_i(\lambda)$, $A_i(u_\lambda)$ and $B_i(u_\lambda)$ for $i = 2, \dots, \nu(\lambda)$ as in case A, with $\nu(\lambda)$ satisfying conditions equivalent to (5.2), (5.3) and (5.12), replacing $\lambda \rightarrow \infty$ by $\lambda \rightarrow 0$ (with $\lambda < 0$). Hence, we assume that $\nu(\lambda)$ satisfies the conditions

$$\lim_{\lambda \rightarrow 0} \nu(\lambda) = +\infty, \quad (5.23)$$

$$\lim_{\lambda \rightarrow 0} \nu(\lambda)^{-1/d} r_\lambda = +\infty, \quad (5.24)$$

and

$$\lim_{\lambda \rightarrow +0} \nu(\lambda) |\lambda|^{-\frac{m+2}{2m}} = 0. \quad (5.25)$$

Proposition 5.5

In Case B, if $\nu(\lambda)$ satisfies (5.23) and (5.24), then we have the following inequality, as $\lambda \rightarrow 0$ ($\lambda < 0$),

$$\sum_{i=2}^{\nu(\lambda)} \#A_i(u_\lambda) \leq \gamma(d) W(\lambda) \left(1 + \mathcal{O}\left(\frac{1}{\nu(\lambda)}\right)\right). \quad (5.26)$$

The proof is the same as in case A. For $W(\lambda)$, we can use the asymptotics (4.11).

For the cardinalities of the sets $B_i(u_\lambda)$, (5.13) holds in case B under condition (5.25) and we can use Corollary 3.8 together with (4.11).

The treatment of the singularity is slightly easier in this case. We use Proposition 3.4 to make a partition of $D_1(\lambda)$ in two annuli:

$$D_{11}(\lambda, C) := \{x \in \mathbb{R}^d \mid r_d < r < C\},$$

and

$$D_{12}(\lambda, C) := \{x \in \mathbb{R}^d \mid C < r < r_\lambda / \nu(\lambda)^{\frac{1}{d}}\}.$$

Again, we choose C such that ν is strictly increasing for $|x| > C$. Since $\lambda < 0$, there exists $M > 0$ such that $\#A_{11}(u_\lambda) < M$. To give an upper bound on $\#A_{12}(u_\lambda)$, we follow the same steps as in case A.

The behavior of the number of eigenvalues below or equal to λ is given by (4.17), hence inequality (5.22) is verified. Since $N(\lambda_n) \leq n - 1$, we obtain Theorem 1.6 in this case as well.

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